Integers and Divisibility

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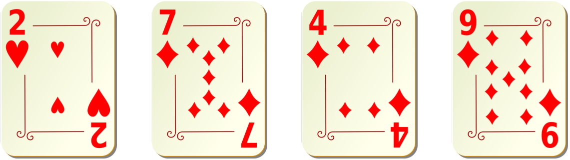
Empirical Inference Department
Online Tea Talk

9 April 2020

Bet with Me



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- Me: I bet 50 EUR that you can't. Want to try?

Should you accept the challenge?

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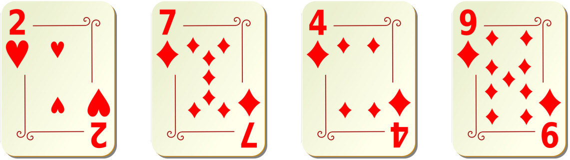

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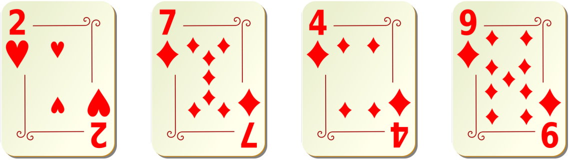

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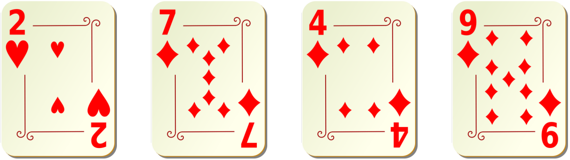

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You (thinking): 7294, 4729, 4792, 9742, ...

Divisibility Tricks

- You should **NOT** take the challenge.
- $\{2, 7, 4, 9\}$ cannot be rearranged to be divisible by 3.
- How can we (mentally) determine this quickly?

In this talk:

> Given some integers F, A,
> ~~discuss how to trick your friends~~
> discuss quick methods to determine whether A is divisible by F.

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Preliminary I

- Let $A \in \mathbb{Z}$ (integers) with n digits.
  Write $A = a_{n-1} \cdots a_1 a_0$ where $a_i \in \{0, \ldots, 9\}$.
- Example: for $A = 267$, $a_2 = 2$, $a_1 = 6$, and $a_0 = 7$.
- Unique decomposition: $A = a_{n-1}10^{n-1} + \cdots a_2 10^2 + a_1 10 + a_0$
- Example: $A = 1369 = 1 \cdot 10^3 + 3 \cdot 10^2 + 6 \cdot 10 + 9$.

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Preliminary II

- Write $D|A$ to mean "D divides A evenly" i.e., $\frac{A}{D}$ is an integer.

Lemma 1

Let $A, B \in \mathbb{Z}$. Let $D \in \mathbb{Z}\backslash\{0\}$. If $D|A$ and $D|B$, then $D|(A+B)$. This generalizes to more than two summands.

- Example: $3|9$ and $3|6$. So $3|(9+6) \equiv 3|15$.

Lemma 2

Let $C := A + B$. If $D|C$ and $D|A$, then $D|B$.

Proof.

First note that if $D|A$, then $D|-A$. We have $B = C + -A$. Lemma 1 implies that $D|B$. □

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Divisibility by 3

- $A = a_{n-1}10^{n-1} + \cdots + a_1 10 + a_0$
- Let $S(A) := \sum_{i=0}^{n-1} a_i = a_{n-1} + \cdots + a_1 + a_0$ (digit sum).

Proposition 3

$3|A$ *if and only if* $3|S(A)$.

Proof.

Consider

$$A - S(A) = a_{n-1}(10^{n-1} - 1) + \cdots + a_2 99 + a_1 9.$$

So,

$$A = \underbrace{a_{n-1}(10^{n-1} - 1) + \cdots + a_2 99 + a_1 9}_{\text{always divisible by 3}} + S(A).$$

If $3|S(A)$, then $3|A$ by Lemma 1. □

- Independent of the order of the digits.
- **The bet:** $S(2749) = 22$ which is not divisible by 3...

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- $T_7(A) := a_{n-1} \cdots a_1 - 2a_0$

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$7 | A$ if and only if $7 | T_7(A)$.

- This suggests a recursive algorithm:
  $7|A \iff 7|T_7(A) \iff 7|T_7(T_7(A)) \iff \cdots$.
- Rina Zazkis, 2002 calls it the **Trimming Algorithm**.

Example: $A = 86415$

A	$T_7(A)$	$=$
86415	$8641 - 2 \cdot 5$	8631
8631	$863 - 2 \cdot 1$	861
861	$86 - 2 \cdot 1$	84

- So, $7|86415$ since $7|84$.

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Note

$$\begin{aligned} A &= 10a_{n-1} \cdots a_1 + a_0 \\ &= 10a_{n-1} \cdots a_1 - 20a_0 + 20a_0 + a_0 \\ &= 10(a_{n-1} \cdots a_1 - 2a_0) + 21a_0 \\ &= 10T_7(A) + 21a_0. \end{aligned}$$

Since $7|21$, if $7|T_7(A)$, Lemma 1 guarantees that $7|A$. □

- But where does 2 come from?

Proof: Divisibility by 7

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A General Trimming Algorithm

Given a prime number $p \notin \{2, 5\}$, there exists $k \in \mathbb{Z}$ such that

$$p | A \iff p | \left[T_p(A) := a_{n-1} \cdots a_1 - k a_0 \right].$$

$$\begin{aligned} A &= 10 a_{n-1} \cdots a_1 + a_0 \\ &= 10 a_{n-1} \cdots a_1 - 10ka_0 + 10ka_0 + a_0 \\ &= 10 \left(a_{n-1} \cdots a_1 - ka_0 \right) + a_0(10k + 1) \\ &= 10 T_p(A) + a_0(10k + 1). \end{aligned}$$

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Find k such that $p|10k+1$

Given p, $p|10k+1$ if there exists $m \in \mathbb{Z}$ (quotient) such that

$$mp = 10k + 1$$
$$\iff mp \text{ is close to } 10k \text{ by } 1$$

Write $p = p_{l-1} \cdots p_1 p_0$. Since $p \notin \{2,5\}$ is prime, we know $p_0 \in \{1,3,7,9\}$.

- Case 1: $p_0 \in \{1,9\}$. Finding k is easy.
  - Example: $p = 29$.
  - Choose $m = -1$ and $k = -3$
  - so that $(-1)p - 10(-3) = 1$.

  Similarly for $p_0 = 1$.
- Case 2: $p_0 \in \{3,7\}$. Reduce it to Case 1.
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- When $p = 7$, choose $m = 3$ and $k = 2$.

Find k such that $p|10k+1$

Given p, $p|10k+1$ if there exists $m \in \mathbb{Z}$ (quotient) such that

$$mp = 10k + 1$$
$$\iff mp \text{ is close to } 10k \text{ by } 1$$

Write $p = p_{l-1} \cdots p_1 p_0$. Since $p \notin \{2, 5\}$ is prime, we know $p_0 \in \{1, 3, 7, 9\}$.

- Case 1: $p_0 \in \{1, 9\}$. Finding k is easy.
  - Example: $p = 29$.
  - Choose $m = -1$ and $k = -3$
  - so that $(-1)p - 10(-3) = 1$.

  Similarly for $p_0 = 1$.
- Case 2: $p_0 \in \{3, 7\}$. Reduce it to Case 1.
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Other Tricks

- Divisibility by 2, 5, 10 is obvious.
- Divisibility by 6 $\implies$ $6 = 2 \times 3$. So, do 2-test and 3-test.
- Divisibility by 9 $\implies$ Same as 3-test.
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More information:

- Numberphile: https://www.youtube.com/watch?v=yi-s-TTpLxY
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Questions?

Thank you

Divisibility by 11

- $A = a_{n-1}10^{n-1} + \cdots + a_2 100 + a_1 10 + a_0$
- Let $C(A) = \sum_{i=0}^{n-1}(-1)^i a_i = \cdots - a_3 + a_2 - a_1 + a_0$

Proposition 6

$11|A$ if and only if $11|C(A)$.

Proof.

$$A - C(A) = a_1 11 + a_2 99 + a_3 1001 + a_4 999 + \cdots$$

$$= \sum_{\text{even } i}^{n-1} a_i \left[10^i - 1\right] + \sum_{\text{odd } i}^{n-1} a_i \left[10^i + 1\right]$$

- $\{10^i - 1 : \text{even } i \geq 2\} = \{99, 9999, 999999, \ldots\}$. All divisible by 11.

$\left\{10^i + 1 : \text{odd } i\right\} = \{11, 1001, 100001, \ldots\}$

$= \{11, 990 + 11, 99990 + 11, \ldots\}$ (All divisible by 11)

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$4 \mid A$ *if and only if* $4 \mid a_1 a_0$.

Example: 836320 is divisible by 4 because 20 is divisible by 4.

- Sufficient to check divisibility on the two least significant digits.

Proof.

$$A = \underbrace{a_{n-1}10^{n-1} + \cdots + a_2 10^2}_{:=X} + \underbrace{a_1 10 + a_0}_{:=Y} = X + Y.$$

$(\Leftarrow)$:

- Each term in X is divisible by 4 regardless of $a_{n-1}, \ldots, a_2$.
- If $Y = a_1 a_0$ is divisible by 4, then the Lemma 1 guarantees $4 \mid A$.

$(\Rightarrow)$: Use Lemma 2 □

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